Lecture 7 - Construct and Compare the Natural Integers

Fabienne Chaplais

March 11, 2026

1 Introduction

The natural integers are the numbers we count with, plus 0. That definition is formalized in the so called 'Peano Axioms', that construct the set of natural integers as a line beginning with 0 and with no end.

Comparing the positions of the numbers on that line leads to define the comparison relations, defining $\mathbb{N}$ as an increasing sequence with minimum 0 and no maximum (the 'greatest possible number' doesn't exist).

A way to compare two 'regular' numbers, the natural integers in 10 basis. is then displayed, the proof being postpone to lecture 37, when we will know more about the decimal system of numeration.

And, last but not least, the appendix is a comprehensive presentation of a powerful way of proving theorems, the so called proof by recursion.

2 Construction of the Set of Natural Integers

We will see two ways of building the set of natural integers, the numbers we count with.

2.1 Naive Construction: counting further and further

The set of the natural integers, or more precisely the set $\mathbb{N}^* = \{1, 2, 3, \dots\}$, is built naively by counting.

It begins with 1 and continues with the sequence of all "regular" numbers.

There is no end to that process, because when we have any number of objects, we can always add another object, or continue to walk on the line.

That naive construction may be formalized with the Peano Axioms.

2.2 Rigorous construction: the Peano Axioms

The Peano Axioms are the basic statements on which the construction of the set of natural integers relies.

The Peano Axioms rely on the definition of the minimum natural number 0 and of the notion of "follower" of a natural integer. The latter notion is the action of counting.

They are four of them:

- 0 is a natural integer:
- The follower of any natural integer is another natural integer
- 0 is not the follower of any natural integer (no loops)
- Any non-zero natural integer is the follower of a unique natural integer (no branches)

These axioms construct the set of natural integers, that is a line because of axioms 3 and 4.

The line starts from 0 (axioms 1 and 3) and continues permanently with iterative process of finding the follower, process that is internal to the set of natural integers because of axiom 2.

3 Comparison of two natural integers

We shall now see how to compare two natural integers.

3.1 Ordering the natural integers line

The natural integer *a is* strictly lower than the natural integer *b (*$a<b$) if they are not the same number *(*$a\neq b$*) and there exists a sequence of processes "find the follower" from* a to *b.*

In other words, $a<b$ if and only if *a is situated to the left of* b on the natural integers line.

For instance, 0 is strictly lower than any non-zero natural integers, because 0 is at the very beginning of the natural integers line.

3.2 Strict comparison of numbers

We can compare is a strict sense any two natural integers, *provided that they are different.*

Let us then consider two natural integers a and *b, that are so that* $a\neq b$.

Then, $a<b$ *if and only if a is before* b on the natural integers line (as we already saw).

And what is *a is NOT strictly lower than* b ?

That means that *a is not to the left of* b on the line.

So, as it is a line, and as $a\neq b$, a is to the right of *b on that line.*

Then, it is said that a is *strictly greater than* b, which is denoted $a>b$.

Theorem 1*:If a and b are two different natural integers, then:*

- either $a<b$, and then $b>a$,
- *or* $a>b$*, and then* $b<a$*.*

Proof

Let's consider two natural integers *a and* b so that $a\neq b$*, and let's consider their placement on the natural integers line: either* a is to the left of *b, and then* $a<b$, or *a is to the right of* b, that is $a>b$.

***Let's now suppose that* $a<b$. Then *b is to the right of* a, so that $b>a$.**

***And if* $a>b$, then *b is to the left of a, that is* $b<a$. QED.**

3.3 Non strict Inequalities

In contrast with the strict inequalities *a<b or* a>b, the non strict comparison operators *and* ≥ allow the comparison of any pair of natural integers, regardless if they are different or equal.

The natural integer *a is said to be less or equal to* b, that is denoted *ab, if and only if either* a<b or *a=b.*

The natural integer a is said to be greater or equal to *b, that is denoted* a≥b, if and only if either *a>b or* a=b.

Theorem 2: If a and b are two natural integers, equal or different to one another, then:

- *ab if and only if a is NOT strictly greater than b.*
- a≥b if and only if a is NOT strictly less than b.

Let's first prove the first direction of the equivalence of the first bullet:

$a \leq b \Rightarrow a \not> b$

Let's consider *a and* b two natural integers so that $a \neq b$.

Then either $a = b$ or a is to the left of b .

Consequently, a is not to the right of b .

The implication (1) is thus proved.

Let's now prove the reverse direction of the first bullet:

$a \not> b \Rightarrow a \leq b$

Then either *a* $\neq$ *b, and then, as* $a \not> b$ *,* $a < b$ *because of Theorem 1, and thus* $a \leq b$ *.*

The implication (2) is thus proved.

Let's now prove the first direction of the equivalence of the second bullet:

$a \geq b \Rightarrow a \not< b$

Then either $a = b$ *or* a *is to the right of* b *.*

Consequently, a *is not to the left of* b *.*

The implication (3) is thus proved.

Let's now prove the reverse direction of the second bullet:

$a \not< b \Rightarrow a \geq b$

Then either a $\neq$ b, and then, as $a \not< b$, $a > b$ because of Theorem 1, and thus $a \geq b$.

The implication (4) is thus proved.

3.4 The totally ordered set of natural integers

Theorem 3: The relation is an "order relationship" in $\mathbb{N}$ *because it has the following properties.*

1. It is reflexive aa for any natural integer a
2. *It is anti-symmetric: if ab and ba, then a=b*
3. It is transitive: if ab and bc, then a≤c

This order relationship is a total order, because any two natural integers a and b may be compared to each other, that is that at least ab or ba. If a=b, both are true, and it is the only case where both are true.

Proof

1. Reflexivity of $\leq$

Let's consider one natural integer *a.*

Then a=a, are thus *a≤a, and ≤ is reflexive*

2. Anti-symmetry of $\leq$

Let's consider two natural integer *a and* b, so that *a≤b and* b≤a.

As *a≤b, we have, by definition of the relation ≤: either* a<b or *a=b.*

And as ba, wee have, thanks to Theorem 2: $a \not> b$ *.*

But a<b means *b>a, which is contradictory with* $a \not> b$ *.*

Thus a=b whenever *a≤b and* b≤a, and ≤ is anti-symmetric.

3. Transitivity of $\leq$

Let's consider three natural integer *a,* b and *c, so that* a≤b and *b≤c.*

As a≤b, we have, by definition of the relation ≤: either *a<b or* a=b.

And as *b≤c, we have also: either b<c, or b=c.*

So that we have the 4 combined following possible cases:

1. a=b and *b=c: then* a=c, and thus *a≤c*

2. a<b and *b=c: then we may substitute* c to *b in* a<b, so that *a<c, and thus* a≤c.

3. *a=b and* b<c: then we may substitute *a to* b in *b<c, so that* a<c, and thus *a≤c.*

4. a<b and *b<c: then* a is to the left of *b on the natural integer line, and* b is to the left of *c on that line, so that* a is to the left of *c and* a<c, and thus *a≤c.*

We see that we have a≤c in all the 4 cases.

Thus *a≤c whenever* a≤b and *b≤c, and ≤ is transitive.*

In conclusion, ≤ is reflexive, anti-symmetric and transitive, so that it is an order relationship.

Let's now prove that is is a total order.

Let's consider a and *b two natural integers.*

Then either a=b, and then *a≤b and* b≤a, or *a≠b.*

If a≠b, then thanks to Theorem 1, either *a<b, and then* a≤b, or *a>b, and then* b<a, so that *b≤a.*

The two last assertions of Theorem 3 are thus proved. QED.

Corollary: $\mathbb{N}$ *is an increasing sequence with the minimum 0 and no maximum.*

1) $\mathbb{N}$ *is an increasing sequence because the follower of any natural integer* n is greater than *n. Indeed, is to the right of* n on natural integer line, by construction of that line.

2) As 0 is at the very beginning of the line, is is less or equal to any natural integers. As it is a natural integer (Axiom 1), it is the minimum of all the natural integers.

3) Let's suppose that has a maximum M, that is a natural integer. Then the follower of *M is greater than* M, that is a contradiction with the fact that *M is a maximum. QED*

Note that in fact, the sequence of increasing natural integers goes to infinity, but this point is out of the scope of the present note.

4 Compare two natural integers displayed in the 10 basis

We shall now deal with a practical issue, how to compare two 'regular' numbers, that are natural integers given in the 10 basis.

4.1. The digits comparison

The digits in 10 basis are the numbers from 0 to 9.

The ordering of these numbers is: $0 < 1 < 2 < 3 < 4 < 5 < 6 < 7 < 8 < 9$ ***, because they are the first 10 numbers on the natural integer line, and because this line is an increasing sequence.***

So that you can see for instance that 3<7.

4.2. The digit by digit process

If two natural integers are represented in the 10 basis, they may be compared the following way:

1. ***If*** **a has less digits than *b, then* a<b (example 23<132).**
2. **If a and be have the same number of digits and the digit to the left of *a is lower than the digit to the left of* b, then *a<b (example* 132<243*).***
3. ***If the number of digits are the same and the*** **k first digits to the left are the same, but the digit number *k+1 of a is lower than the digit number* k+1 of *b, then a<b (examples 1*23<132, and 132<135).**
4. **If a has all its digits equal to the correspondent digits of *b, then* a=b.**
5. **If both *a>b and* a=b are false, then *a>b.***

The proof of that process will be done by recursion (see Appendix) in Lecture 44, "Comparison Scheme in Decimal Notation".

5. Conclusion

We have built the totally ordered line of the natural integers, with the relationship 'less or equal to'.

On that line, any two numbers may be compared and, if they are different, they may be compared in one direction only.

Then we end the note giving the practical way to compare two 'regular' integers given in 10 basis.

5 Appendix-The proof by recursion-

5.1 Introduction

The construction of the natural integers set allows a very powerful proving method, the proof by recursion, also called the proof by induction.

We shall define it formally in that text, giving a rigorous justification of such a proof process.

We will end the appendix with the alternative versions of the proof by recursion, that give to it an increased power.

5.2 Definition of the proof by recursion

The proof by recursion of an assertion depending of an index follows the following process:

- **InitializationThe property P_n is true for $n = 0$ (P_0 is true)**
- **RecursionUnder the hypothesis that P_n is true, for a given $n \in \mathbb{N}$, then P_{n+1} is true for the follower $n + 1$ of n .**

If these steps are proved, then the property is true for any $n \in \mathbb{N}$.

5.3 Justification of the proof by recursion

The proof by recursion is justified by the Peano axioms given in the paragraph 2.2.

Namely, it is the axiom number 4, "Any non-zero natural integer is the follower of a unique natural integer", that gives the main part of the justification.

Indeed, if the initialisation in $n = 0$ is justified by the axiom number 1, "0 is a natural integer", the core of the process is the recursion by itself.

And the fact that any non-zero natural integer is the follower of a unique natural integer allows to prove P_n after we have proved P_{n-1} for the previous natural integer (n is the follower of $n - 1$).

And we may continue like that from the right to the left of the natural integer line, until we meet 0, for which the property is proved during initialisation.

And we meet effectively 0, because any the natural integer is built by a finite processes of "find the follower", starting from 0.

5.4 Alternate versions of the proof by recursion

We give here some alternate versions of the proof by recursion, may that be for initialization, of for recursion itself.

5.4.1 Initialisation variants

You may initialise your proof by recursion at a rank n_0 ***. In such a case, your property will be proved only for the natural integers*** $n \geq n_0$ ***.***

You may also initialize it with several conscutive initial values, for $n_0, n_0 + 1, \ldots, n_0 + k$ ***. Then your property will be proved also for the natural integers*** $n \geq n_0$ ***.***

5.4.2 Recursion main variant

The recursion process may be replaced by:

If P_m ***is true for any m*** **so that** $n_0 \leq m \leq n$ ***, then*** P_{n+1} ***is true.***

This is a very powerful variant, because, for the proof of P_{n+1} ***, we may rely on all the assertions*** P_m ***, for all the successive*** m ***so that*** $n_0 \leq m \leq n$ ***.***

5.5 Conclusion

As you will see in the rest of the course, in the other courses of that series, and more generally in your practice of math, the proof by recursion is awfully useful.